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The effects of a strongly temperature-dependent viscosity on slow flow past a hot sphere

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This work determines analytically the drag on, and heat flux out of, a hot sphere that translates steadily in a fluid of strongly temperature-dependent viscosity. There is no dissipative heating. The essentials are illustrated by an exact solution for the flow induced by slowly squeezing two parallel planes together. The lower plane is hot and stationary; the upper is cold and advances in a direction normal to itself at uniform speed U. The gap is completely filled by a fluid of strongly temperature-dependent viscosity. We find the temperature and velocity profiles, and determine the Nusselt number N and Péclet number P as functions of the normal force D on the lower plane. The large viscosity variation tries to concentrate the flow into a relatively thin softened layer in which the viscosity is of order its value μ_0 at the hot plane. In the limit of infinite viscosity ratio (fixed P), it succeeds (lubrication limit): if $P \ll 1$, the width of the softened layer is determined by conduction and $D \propto \mu_0 U$; but $D \propto \mu_0 U^4$ when forced convection is important. If $P \rightarrow \infty$ (fixed viscosity ratio), the softened layer is so thin that it chokes, and all the deformation occurs outside the thermal layer in the fluid of uniform viscosity μ_∞ (Stokes limit); then $D \propto \mu_\infty U$. These mechanisms appear as three distinct legs in our plot of $\log P$ against $\log D$. There are similar transitions in the plot of $\log N$ against $\log D$. The solution gives an estimate of the drag on a sphere. We test this estimate against an analytical solution for the sphere in the lubrication limit. Then we extend the solution to cover power-law fluids, and apply it to a model (by Marsh) of magma transport beneath island-arc volcanoes. The results suggest that the magma covers the first 50 km of its ascent by an isoviscous mechanism, with the lubrication mechanism operating in the remaining 50 km. To open a fresh pathway from the source to the surface takes about 10^6 years and uses about 10^{27} erg.

1. Introduction

A number of geophysical flows are very strongly influenced by the extreme sensitivity of the viscosity to temperature changes. The Prandtl number of the fluid is effectively infinite in all these cases. In this, and in some papers to follow, we shall exploit this strong dependence on temperature to give analytical solutions to some model problems that illustrate the essential features of these variable-viscosity flows.

In this paper we determine the steady creeping flow of an incompressible fluid past a stationary hot sphere of radius a and constant temperature T_0. Far from the sphere the flow is uniform, with prescribed velocity and temperature T_∞. We suppose the temperature field to be solely the result of a balance between the conduction of heat away from the sphere and the forced convection of cold fluid towards it. Because the viscosity depends on temperature, the momentum and energy equations must be solved simultaneously. Except in the conduction limit, the problem is thus nonlinear, and the point of the asymptotic method is that, when the viscosity depends very strongly on temperature, the momentum and energy equations can be solved sequentially. This allows an analytical solution to be found.

This problem is the analogue for a variable-viscosity fluid of Stokes's problem. We shall use the drag law for the sphere to find the terminal speed of a sphere that is lighter than the surrounding fluid because of an imposed density difference. The problem was originally posed by Marsh (1978) as a model of magma transport beneath island-arc volcanoes. Magma is generated about 100 km down. Petrologists face the difficulty that the Earth's lithosphere is so stiff that, to reach the surface, a sphere of radius 1 km (say) following Stokes's law would need a time comparable to the Earth's age. Marsh recognized that, because the magma is hotter than its surroundings for most of its ascent, it will lower the viscosity of the fluid near it and rise more quickly than Stokes's law would allow.

It turns out that the variable-viscosity flow transfers heat very effectively. A sphere rising with only its primal heat cools to the temperature of its surroundings once it has travelled through one or two of its own diameters. For the softening mechanism to work, either successive magmas must use the same pathway, or else there must be a trailing stalk which feeds fresh, hot magma to the sphere. The rapid heat loss is a nuisance for an individual parcel; but it is essential for the process as a whole, which relies on pumping heat from the source region into the cold country rock just ahead of the advancing hot front of the diapir.

Once a pathway has been opened, there will be little difficulty in getting magma to the Earth's surface. Thus we shall estimate the time taken to open a fresh pathway through virgin rock by using the results for the steady motion of a sphere at constant temperature T_0. We do not treat the problem of maintaining the temperature T_0.

This response time for the island arc is geologically constrained. At any given oceanic trench, subduction has been going on for a certain time. For there to be the observed association between present subduction zones and active island-arc volcanoes, the response time must be less than the age of the youngest subduction zone that has volcanoes associated with it. Using this approach, Marsh (1982) has estimated the response time to be less than about 1×10^6 years. Given enough hot magma, ascent by softening meets this constraint.

In §2 we describe the two essential qualitative features of the flow past a hot sphere. These features are predicted by an asymptotic theory for a fluid with a viscosity profile $\nu(T) = A\,e^{-\gamma T}$ and $\gamma(T_0-T_\infty)\rightarrow\infty$. It is reasonable to expect them to appear in the actual geophysical flow only if the condition $t=\gamma(T_0-T_\infty)\gg 1$ is met. We shall see that the asymptotic theory is qualitatively valid if $t>10$. This condition is met in the geophysical flow. For example figure 1 shows the viscosity of olivine at zero pressure. It was calculated using an activation energy of 522 kJ mol^{-1} (Ashby & Verrall 1978). The viscosity profile is actually of the form $\mu=\mu_*\exp(C/T)$, where $C=6{\cdot}28\times10^4$ K. Near 1500 K, for example, this profile can be approximated by the form $\nu=A\,e^{-\gamma T}$ ($\gamma^{-1}=35{\cdot}8$ K) over a range of about 400 K. This means that $t>10$ if $T_0-T_\infty>360$ K. This condition is met whenever the softening mechanism is important (see §5).

We shall concentrate on the geophysical applications of this study, but there are at least two other applications. The squeezing flow between parallel planes, which we discuss in §3, has immediate applications to the moulding of polymers (see e.g. Lee *et al.* 1982). The problem of the sphere may also have an application to pathology: the pathologist Florey showed that mucus acts as a mechanical barrier to hinder bacterial invasion of the body. It is an effective barrier because mucus is very viscous,

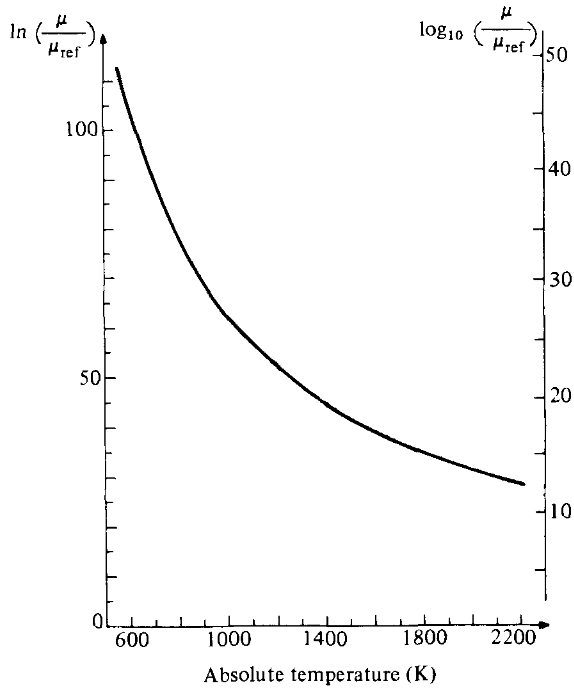

FIGURE 1. Viscosity of olivine at zero pressure as a function of absolute temperature (K). Data from Ashby & Verrall (1978). Activation energy 522 kJ mol^{-1}, $\mu_{\text{ref}} = 2{\cdot}67$ kg m^{-1} s^{-1} at 1000 K.

but bacteria secrete an enzyme that lowers its viscosity. This is simply the chemical version of the problem studied here. The pathology is briefly described by Macfarlane (1979, p. 278).

Finally, it is worth noting that a similar asymptotic method has been used to study both channel flows for polymer processing, and also combustion problems in which the rate of energy release depends strongly upon temperature. See Ockenden & Ockenden (1977) and Pearson (1977) for examples of the first problem, and Clavin & Williams (1979) for an example of the second.

2. Qualitative aspects of the flow

When the viscosity of the fluid depends strongly upon temperature, two essential features emerge: first, the essential viscosity variations occur over a distance l much shorter than the lengthscale δ of the temperature field; secondly, there is an interaction between the large viscosity contrast and the size of the system, which means that even flows with a large viscosity contrast can, in the right circumstances, show qualitatively isoviscous behaviour.

To see why the first of these features occurs, consider the case $\nu(T) = A\,e^{-\gamma T}$; the temperature of the sphere is T_0, the temperature of the undisturbed fluid is T_∞, and $\Delta T = T_0 - T_\infty$. Let $\nu_0 = \nu(T_0)$ and $t = \gamma \Delta T$. Then

$$\nu = \nu_0 \exp t\left(\frac{T_0 - T}{\Delta T}\right),$$

and it follows that, as $t \to \infty$, $\nu/\nu_0 \to \infty$ exponentially, except in the region in which $T_0 - T = O(\gamma^{-1})$. Now the temperature field set up by the hot sphere will have the qualitative form shown in figure 2. Since $\gamma^{-1} \ll \Delta T$, the layer in which $\nu/\nu_0 = O(1)$

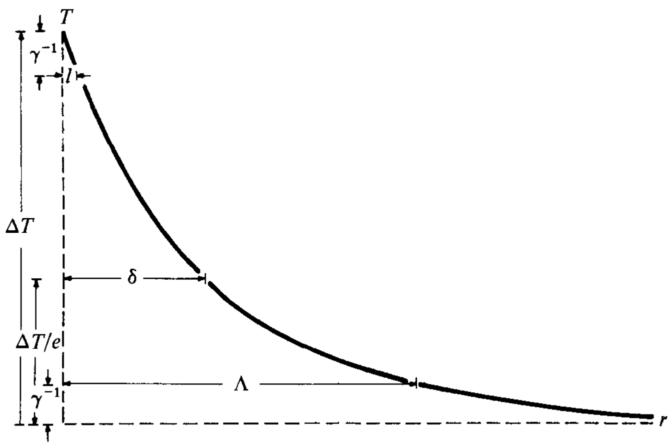

FIGURE 2. Schematic showing the temperature field outside the hot sphere.

is much thinner than the thermal boundary layer. The thickness l of this deformation layer can easily be estimated. For since the temperature difference across it is order γ^{-1}, the temperature gradient within it is of order γ^{-1}/l. But since l is small compared with the thickness δ of the thermal boundary layer, the temperature gradient within the deformation layer is comparable to its value $\Delta T/\delta$ outside the layer. Hence $\delta/l \approx t$, so that $\delta/l \rightarrow \infty$ as $t \rightarrow \infty$.

It is important to recognize that l is the distance needed for the viscosity to increase by a factor e over its value at the surface of the sphere. In fact

$$l^{-1} \approx \frac{\partial}{\partial r} \ln \nu \approx \delta^{-1}\left(\Delta T \frac{d}{dT_0} \ln \nu_0\right).$$

From the definition of this layer, it follows that outside it $\nu/\nu_0 \rightarrow \infty$ exponentially as $t \rightarrow \infty$. This means that the ratio ν/ν_0 must also increase exponentially with distance across the deformation layer in order to match to this large external value; this exponential increase occurs over the lengthscale l. By itself this exponential increase in viscosity will confine the shear to the deformation layer. As we shall now see, substantial velocity differences can appear outside the soft layer only if the region across which they occur is so large that the lengthscale of the flow can compensate for the very large viscosity. This will mean that substantial velocity differences can occur outside the deformation layer only for very large spheres, and only in the isoviscous part of the flow.

The strong temperature dependence of the viscosity thus separates the velocity field into two different parts: the first is the flow in the very narrow but relatively inviscid deformation layer; the second is the isoviscous flow, which occurs outside the thermal boundary layer in the very viscous fluid. The lengthscale of the second flow is of order of the radius a of the sphere.

The second essential feature of these variable-viscosity flows is that under some circumstances the isoviscous flow can be more important than the flow in the softened layer. When that is so, the drag on the sphere is given in order of magnitude by Stokes's law.

To see how this can occur, not that for the sphere to advance through one of its own radii a, it must push behind it a volume πa^3 of fluid. Let all the parameters that describe the flow be fixed, except the viscosity ν_0 in the softened layer. Then, if ν_0/ν_∞ is made small enough, the pressure needed to drive the volume πa^3 of fluid through

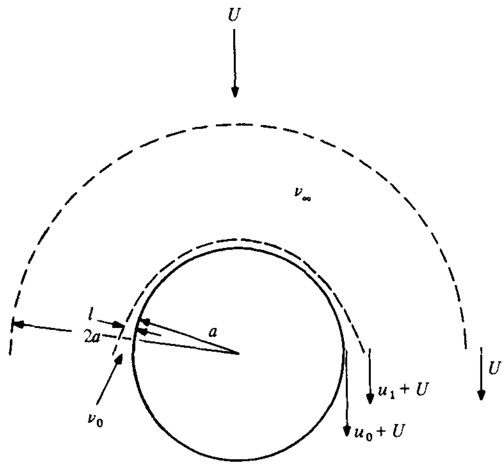

FIGURE 3. Notation for the scaling analysis.

the very narrow softened layer will be much less than that needed to deform the very viscous surrounding fluid. Almost all the shear will then be confined to the width l of the thin softened layer, and the streamlines will look very much like those in the corresponding melting problem.

Suppose now that the ratio ν_0/ν_∞ is fixed at this small value, and the ratio a/l is increased by increasing the Péclet number Ua/κ. Then, because the deformation layer becomes very narrow, the pressure drop needed to drive the volume πa^3 through the thin layer becomes so large that it starts to deform even the most viscous fluid. Increasing a/l beyond this value will result in the volume πa^3 passing the sphere in the isoviscous region, rather than through the deformation layer. For these very large values of a/l, both the drag on, and the heat flux out of, the sphere will be essentially that for an isoviscous flow.

Clearly, we need a method to determine which strategy a given sphere will follow. The following two-layer model tells us the dimensionless parameter that determines this. Figure 3 shows the notation.

Let $(u_r, u_\theta) = (w, u)$. Since the disturbance due to the sphere extends over a distance of order a, conservation of mass requires that

$$aU \approx lu_0 + au_1. \tag{2.1}$$

Since the deformation layer is thin, the pressure gradient in it is of the same order as the pressure gradient in the external flow. Also, if the tangential velocity is much greater in the deformation layer than it is outside, then $\partial u/\partial r \approx u_0/l$ within the deformation layer. Conservation of momentum thus requires that

$$\mu_0 \frac{u_0}{l^2} \approx \frac{1}{a}\frac{\partial p}{\partial \theta} \approx \mu_\infty \frac{u_1}{a^2}. \tag{2.2}$$

Together these equations determine u_0 and u_1. Let

$$\Delta = \left(\frac{a}{l}\right)^3 \frac{\nu_0}{\nu_\infty} = \epsilon N^3, \tag{2.3}$$

where N is the Nusselt number and $\epsilon = t^3(\nu_0/\nu_\infty)$. Then

$$u_1 \approx \frac{\Delta}{1+\Delta} U \approx w_1, \tag{2.4}$$

$$u_0 \approx \frac{(a/l)\,U}{1+\Delta}, \quad w_0 \approx \frac{U}{1+\Delta}. \tag{2.5}$$

The drag on the sphere is of order $(p-p_0)\pi a^2$, where p_0 is the pressure on the equator. From the second equality in (2.2) it follows that the drag

$$D \approx \frac{\mu_\infty a U}{1+\Delta^{-1}}. \tag{2.6}$$

There are clearly two limiting cases. If $\Delta \gg 1$,

$$D \approx D_\infty = \mu_\infty a U,$$

so that the drag is given by Stokes's law in order of magnitude; but, if $\Delta \ll 1$,

$$D \approx D_0 = \mu_0 \left(\frac{a}{l}\right)^3 a U.$$

Note that in the second case the drag increases as l decreases; it is also independent of μ_∞.

Thus there are two mechanisms by which the sphere can move through the fluid. The parameter Δ is simply the ratio D_0/D_∞, and the scaling argument shows that for a given value of Δ the sphere picks the strategy with the smaller drag. This useful conclusion is carefully qualified in §3.3.

The thickness l is determined by the energy equation, and, like the drag, the heat transfer can either be dominated by the flow in the deformation layer, or else by the isoviscous flow. To see the essential point, note that (2.5) shows that the jump in normal velocity across the deformation layer is of order U if $\Delta \ll 1$, but that it vanishes for $\Delta \gg 1$. In the first case the isoviscous flow sees the surface of the sphere as porous, with a suction velocity of order U, and the heat transfer is very efficient. If on the other hand $\Delta \gg 1$, the isoviscous flow satisfies the boundary condition on the normal velocity. But even if the surface of the sphere is rigid there is always a range of values of Δ for which the isoviscous flow sees the surface as stress-free. We shall see that for a solid sphere this means that there are three essential knees in the heat-transfer curve.

Perhaps it will be useful to explain here why the body of this paper discusses squeezing flow between parallel planes. Recall that the simple exact solution for the Stokes problem is possible only because the flow is highly symmetric; the derivation in Batchelor (1967) makes this explicit. For the variable-viscosity problem there is no fore-and-aft symmetry in general and there is no simple exact solution valid for all Δ. For this work I needed a model flow with two properties. First it had to be sufficiently simple for there to be an exact solution for all Δ; this solution had to show the same qualitative behaviour as that for the sphere. Secondly, the solution had to give a reasonably accurate estimate of the drag on the sphere in the most interesting case $\Delta \rightarrow 0$. Squeezing flow between parallel planes meets both criteria.

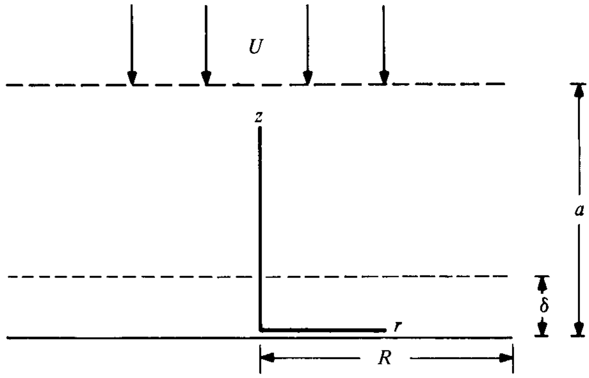

FIGURE 4. Geometry for the hot stagnation-point flow.

3. Squeezing flow between parallel planes for arbitrary Δ and Ua/κ

This is the analogue for a variable-viscosity fluid of the classic problem due to Reynolds; for that problem see Batchelor (1967, p. 228). To solve the problem we first use the fact that $l/\delta \to 0$ to show that the temperature varies linearly across the deformation layer. This lets us write down the viscosity in terms of δ. We can then solve the momentum equation, but, because the solution contains the unknown quantity δ, the resulting drag law (3.20) gives the dimensionless drag in terms of the Nusselt number N. We then solve the energy equation to obtain the relationship between the Nusselt number N and the Péclet number P.

Figure 4 shows the geometry of the flow. Let u and w be respectively the radial and axial components of the velocity. The boundary conditions are that

$$w = -U, \quad u = 0, \quad T = T_\infty \quad \text{on} \quad z = a,$$

and that

$$w = 0, \quad T = T_0 \quad \text{on} \quad z = 0.$$

Also, either

$$u_z = 0 \quad \text{or} \quad u = 0 \quad \text{on} \quad z = 0.$$

We shall not specify whether the plane $z = 0$ is rigid or traction-free until we discuss the drag in §3.2.5; from there on a general formulation is still easy, but it becomes unreadable. We discuss the solution in detail for the traction-free case $u_z(x, 0) = 0$, and merely state the results for the rigid case $u(x, 0) = 0$ in §3.4.

3.1. *Governing equations*

Let $(u, w) = (\frac{1}{2}rf'(z), -f(z))$. Then the equation of continuity is satisfied identically, and the exact momentum equations are

$$\tfrac{1}{2}f'^2 - ff'' = G_0 + [\nu(T)f'']', \tag{3.1}$$

$$ff' = -\frac{1}{\rho}p_z - \nu f'' - 2\nu_z f'. \tag{3.2}$$

Here G_0 is constant and

$$\frac{1}{\rho}p_r = -\tfrac{1}{2}G_0 r. \tag{3.3}$$

Now let $T = T(z)$. Then the exact energy equation is

$$-fT' = \kappa T''. \tag{3.4}$$

Finally, we shall consider the case in which

$$\nu = A\,e^{-\gamma T}, \tag{3.5}$$

although the main conclusions will not depend on the choice (3.5) if separation of scales holds.

From the discussion of the energy equation in §2, it follows that, when external deformation is negligible, the thickness of the thermal boundary layer is of order κ/U. Therefore we use the dimensionless variables

$$\tilde{z} = Uz/\kappa, \quad \tilde{f} = f/U, \quad \tilde{T} = \frac{T-T_\infty}{\Delta T}, \quad \tilde{\nu} = \frac{\nu}{\nu_0}.$$

On ignoring inertial forces, (3.1), (3.4) and (3.5) become

$$0 = \tilde{G}_0 + [\tilde{\nu}\tilde{f}'']', \tag{3.6}$$

$$0 = \tilde{f}\tilde{T}' + \tilde{T}'', \tag{3.7}$$

$$\tilde{\nu} = \exp t(1-\tilde{T}). \tag{3.8}$$

Here we have put

$$\tilde{G}_0 = \frac{\kappa^3}{\nu_0 U^4} G_0, \quad t = \gamma \Delta T.$$

These equations are to be solved subject to the conditions $\tilde{f}(P) = 1$, $\tilde{f}'(P) = 0$, $\tilde{f}(0) = 0$, $\tilde{T}(0) = 1$, $\tilde{T}(P) = 0$, and either $\tilde{f}'(0)$ or $\tilde{f}''(0) = 0$. Here the dimensionless distance $P = Ua/\kappa$ is the Péclet number based on the separation a of the plates. It will be helpful later to notice here that

$$N = \frac{a}{\Delta T}|T_0'| = P|\tilde{T}_0'|. \tag{3.9}$$

Finally we let

$$\Delta = \epsilon N^3, \quad \text{where} \quad \epsilon = t^3 \frac{\nu_0}{\nu_\infty}. \tag{3.10}$$

We may now drop the tildes on dimensionless quantities.

3.2. *The momentum equation*

3.2.1. *The deformation layer.* In §2 we saw that there is a layer near $z = 0$ in which $\nu/\nu_0 = O(1)$ as $t \to \infty$. We now show that within it the viscosity increases exponentially with z. From (3.7) it follows that the Taylor expansion of T about $z = 0$ is

$$T = 1 + zT_0'(1 + \tfrac{1}{6}z^2 w_0' + O(z^3)). \tag{3.11}$$

Now the velocity gradient $|w_0'|$ will be largest when the fluid outside the deformation layer is essentially undeformed, and in that case the estimate (2.5) for the jump in normal velocity shows that $|w_0'| = O(t)$. It follows from (3.11) that, if $z = O(t^{-\frac{1}{2}})$ as $t \to \infty$, then T decreases linearly with z. This justifies the statement that the temperature gradient is uniform across the deformation layer. Thus

$$\nu = \exp t|T_0'|\,z.$$

The parameter $|T_0'|$ is as yet unknown; it will be determined only when the energy equation has been solved.

Now let

$$\zeta = t|T_0'|\,z, \quad f = \frac{1}{t^3|T_0'|^3} G_0 \hat{f}(\zeta).$$

Then if $t \to \infty$ the momentum equation (3.6) becomes within the deformation layer

$$0 = 1 + [e^{\zeta} \hat{f}'']', \tag{3.12}$$

which is to be integrated subject to the condition $0 = \hat{f}(0)$, together with the condition that the solution match to the velocity field within the adjustment layer. Hence

$$\hat{f}(\zeta) = (\hat{f}_0' + \hat{f}_0'' - 1)\zeta + (2 - \hat{f}_0'')(1 - e^{-\zeta}) - \zeta e^{-\zeta}. \tag{3.13}$$

3.2.2. *The isoviscous flow.* Since we expect the isoviscous flow to scale on the separation P of the plates, let $Z = z/P$ and let $f = e^{-t} G_0 P^3 F(Z)$. Then (3.6) reduces to

$$0 = 1 + F''''.$$

Here $F'(1) = 0$ and $F(1) = e^t G_0^{-1} P^{-3}$. Hence

$$F(Z) = F_0 + (\tfrac{1}{2} - F_0'')Z + \tfrac{1}{2}F_0'' Z^2 - \tfrac{1}{6}Z^3, \tag{3.14}$$

where

$$F_0 = F(1) + \tfrac{1}{2}F_0'' - \tfrac{1}{3}. \tag{3.15}$$

3.2.3. *The adjustment layer.* Since $\nu = \exp t(1-T)$ it follows that the viscosity returns to its value e^t in the undisturbed fluid only when $T = O(t^{-1})$. From figure 2 it is clear that the distance Λ over which this occurs is far greater than l. This means that there is no region of overlap between the deformation layer and the isoviscous flow. Thus it will be possible to match the two solutions only if the tangential velocity is independent of z in this 'adjustment' layer. This is the case.

The essential point is that the pressure gradient is the same for each of the three layers. A given layer can therefore support a substantial change in velocity across it either if the layer is very thick, or if its viscosity is very small. The isoviscous flow can support a substantial shear for the first reason, and the deformation layer for the second; but the viscosity in the adjustment layer is at least of order $\exp t^{\frac{1}{2}}$ relative to the viscosity in the deformation layer, whilst the ratio Λ/l depends only on some power of t. The two effects together ensure that the tangential velocity is uniform across the adjustment layer, provided that t is large enough. This argument is easily formalized.

We also want to know how big t should be in practice for the adjustment layer to be negligible. The practical way to determine this is by iteration (see §3.3).

3.2.4. *Matching.* Since the tangential velocity is uniform across the adjustment layer we can match the isoviscous flow directly to that in the deformation layer. In a two-layer model of this flow the normal and tangential velocities, as well as the shear stress, would have to be continuous at the interface. Here the viscosity varies continuously with z, and the corresponding matching condition is that both the constant and linear terms, as well as the shear stress, in (3.13) must match the corresponding terms in (3.14). Carrying this out in the usual way gives

$$F_0 = \frac{2 - \hat{f}_0''}{\epsilon N^3}, \tag{3.16a}$$

$$\tfrac{1}{2} - F_0'' = \frac{\hat{f}_0'' + \hat{f}_0' - 1}{N^2 t^2 e^{-t}}, \tag{3.16b}$$

$$F_0'' = \frac{\hat{f}_0''}{Nt}. \tag{3.16c}$$

These are three equations in four unknowns, $\hat{f}_0'$, $\hat{f}_0''$, F_0 and F_0''. Given one more boundary condition on $\hat{f}$ at $z = 0$, the other three quantities follow. We now consider the stress-free case $\hat{f}_0'' = 0$.

3.2.5. *The case* $\hat{f}_0'' = 0$. From (3.15) and (3.16) it follows that the pressure gradient G_0 satisfies

$$G_0\{1+\tfrac{1}{6}\epsilon N^3\} \underset{t\to\infty}{\sim} \tfrac{1}{2}t^3N^3P^{-3}. \tag{3.17}$$

Moreover, the composite expansion for f is

$$f \underset{t\to\infty}{\sim} (1+\tfrac{1}{6}\epsilon N^3)^{-1}\{[1-(1+\tfrac{1}{2}\zeta)\,e^{-\zeta}]-\tfrac{1}{12}\epsilon N^3(Z^3-3Z)\}. \tag{3.18}$$

Here we have used the fact that $N = P|T_0'|$ and the definition $\epsilon = t^3(\nu_0/\nu_\infty)$. Note that, if $\epsilon N^3 \ll 1$, the stream function reduces to

$$f \sim 1-(1+\tfrac{1}{2}\zeta)\,e^{-\zeta},$$

and the shear is confined to a narrow layer of thickness $aN^{-1}t^{-1}$. If on the other hand $\epsilon N^3 \gg 1$,

$$f \sim \tfrac{1}{2}(3Z-Z^3),$$

and the shear fills the fluid. Figure 5 shows the tangential velocity profile for these two extremes.

3.2.6. *The drag law*. The stream of fluid exerts a force on the surface $z = 0$ which is equal to the surface integral of the normal stress. To calculate the total normal force we suppose that the flow has radial extent R; later we shall identify R with the radius a of a sphere. The contribution made by the normal viscous stresses is $(l/R)^2$ times that of the pressure, and it can be ignored.

In §4 we shall find that when a sphere moves by softening its surroundings, the pressure is an order of magnitude less at the equator than it is at the forward stagnation point. Thus we suppose that p vanishes at $r = R$.

On using (3.3) and (3.16) to find the pressure, it follows that the (dimensional) normal force acting on a disk of radius R is

$$D' = \tfrac{1}{16}\pi\mu_0\frac{R^4}{a^3}\,Ut^3N^3\,(1+\tfrac{1}{6}\epsilon N^3)^{-1}. \tag{3.19}$$

Define the dimensionless drag D by

$$D' = \tfrac{1}{16}\pi\mu_0\,\kappa\left(\frac{R}{a}\right)^4 t^3D.$$

Then the drag law is
$$PN^3 = D(1+\tfrac{1}{6}\epsilon N^3). \tag{3.20}$$

Here the Péclet number $P = Ua/\kappa$, and the Nusselt number N is defined in (3.9). Equation (3.20) closely resembles equation (2.6) of the scaling analysis. Given D, there are two unknowns, P and N. As we did in the scaling analysis, here too we shall obtain a second relationship between these unknowns from the energy equation.

3.3. *The energy equation*

Given the velocity profile (3.18), we can solve the energy equation to find both the temperature profile and the relationship between N and P. Even when P is arbitrary we need not use the full profile (3.18). To see this, first consider the case $P \gg 1$ so that the thermal-boundary-layer approximation holds. Then within the thermal layer $\zeta \gg 1$ and $Z \ll 1$, so that

$$f \sim (1+\tfrac{1}{6}\Delta)^{-1}\,(1+\tfrac{1}{4}\epsilon N^3Z) \tag{3.21}$$

within the thermal boundary layer; the first of the two terms is the jump in normal velocity across the deformation layer and the second reflects deformation within the external flow.

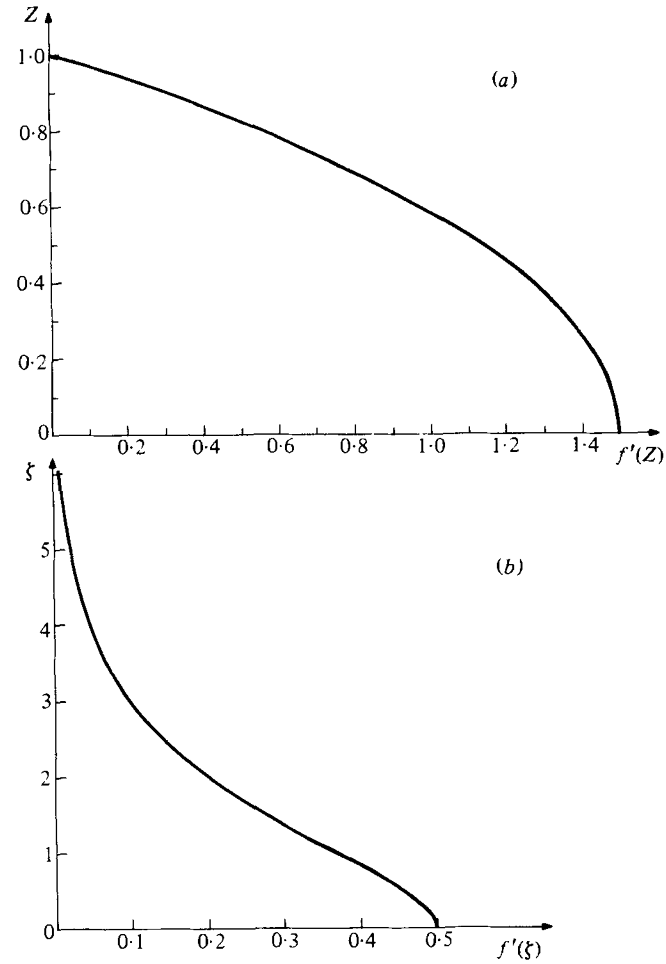

FIGURE 5. Tangential velocity profile for the hot stagnation-point flow: (*a*) Stokes limit; (*b*) lubrication limit.

If, on the other hand, P is less than or of order unity then N is of order unity, and since $\epsilon \to 0$ the appropriate form of (3.18) is $f \sim 1$. Equation (3.21) contains this case, and is thus the most general form of f needed to solve the energy equation.

The temperature profile is given by

$$T-1 = T_0' \int_0^z ds \exp\left\{-(1+\tfrac{1}{6}\Delta)^{-1}\left(s+\frac{\epsilon N^3}{8P}s^2\right)\right\}, \tag{3.22}$$

where $\Delta = \epsilon N^3$. We discuss this later (see figure 8). Here the important point is that the quantity $|T_0'| = N/P$ is fixed by the condition that $T = 0$ when $z = P$. This gives the relationship between N and P. Combining that result with the drag law (3.20) gives a useful relationship between N and D. It is

$$D = N^4 \int_0^{DN^{-3}} ds \exp\left\{-\left(s+\frac{\epsilon N^6}{8D}s^2\right)\right\}. \tag{3.23}$$

Figure 6 shows this $N(D)$ relationship. If ϵ is small enough, each curve has three distinct legs. Table 1 gives their analytical forms. To interpret them, first consider

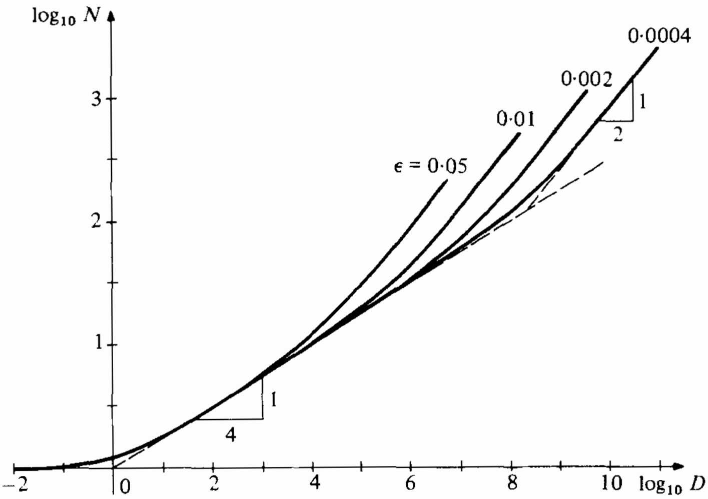

FIGURE 6. Nusselt-number–drag relationship for the hot stagnation-point flow. Dimensional drag $D' = \frac{1}{16}\pi\mu_0\kappa\,(R/a)^4\,t^3 D$.

	Drag D'	$N(D)$	$N(P)$
$0 \leqslant D \leqslant 1$ (Conduction limit)	$\frac{1}{16}\pi\mu_0 \dfrac{R^4}{a^3} t^3 U$	1	$\dfrac{P}{1-e^{-P}}$
$1 \ll D \ll \epsilon^{-\frac{4}{3}}$ (Lubrication limit)	$\frac{1}{16}\pi\mu_0 \dfrac{R^4 t^3}{\kappa^3} U^4$	$D^{\frac{1}{4}}$	$\left(\dfrac{6P}{\epsilon}\right)^{\frac{1}{4}}$
$\epsilon^{-\frac{4}{3}} \ll D \ll \epsilon^{-2}$	$\frac{3}{8}\pi\mu_\infty \dfrac{R^4}{z_3} U$		
$\epsilon^{-2} \ll D$ (Stokes limit)		$\left(\dfrac{\epsilon D}{2\pi}\right)^{\frac{1}{2}}$	$\left(\dfrac{3P}{\pi}\right)^{\frac{1}{2}}$

TABLE 1. Asymptotic forms for the case $f_0'' = 0$ (traction-free)

the knee at $D \approx \epsilon^{-2}$. In that case table 1 shows that $N \approx \epsilon^{-\frac{1}{2}}$, and a thermal boundary layer is present. At the outer edge of this thermal layer $Z \approx N^{-1}$, so that from (3.21) the normal velocity there is

$$\frac{1+\frac{1}{4}\epsilon N^2}{1+\frac{1}{6}\epsilon N^3}. \tag{3.24}$$

If $\epsilon N^2 \gg 1$, this reduces to $\frac{3}{2}N^{-1}$: here the normal velocity at the outer edge of the thermal layer is independent of ϵ, so that the heat transfer is not affected by the presence of the deformation layer. This is the right leg in figure 6. On the middle leg, $\epsilon N^2 \ll 1$ and the thickness of the thermal layer is controlled by the jump in normal velocity across the deformation layer (see (3.21)). At the knee which joins these two legs, the coefficient of s^2 in (3.23) (namely $\epsilon N^6/8D$) is of order unity. Well to the left of the knee we can set this coefficient equal to zero and evaluate (3.23). The

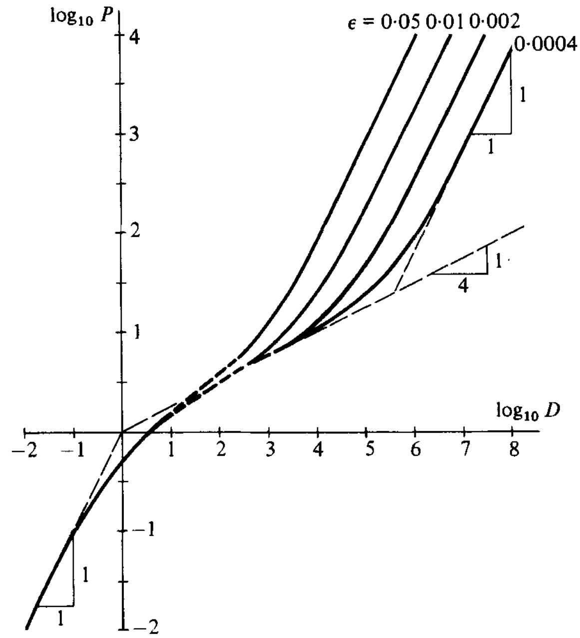

FIGURE 7. Péclet-number–drag relationship for the hot stagnation-point flow.

resulting analytical expression describes the left and middle legs in figure 6, together with the knee that joins them. This knee corresponds to the formation of a thermal boundary layer; well to the left of it $N = 1$, and to the right of it $N \gg 1$.

At the right knee $DN^{-3} \approx \epsilon^{-\frac{1}{2}}$ and is large when ϵ is small. This means that, if $\epsilon \to 0$, the right half of the graph can be generated by setting $DN^{-3} = \infty$ in (3.23). The left half is generated by setting $\epsilon N^6/8D = 0$ in (3.23). The values of ϵ I considered were small enough for the two halves of the $N(D)$ curve to match. But the same method, applied to the $P(D)$ curves in figure 7, gives a visible mismatch for the largest values of ϵ. In the overlap region this corresponds to an error in P of about 5 %. The parts of the curve with a visible mismatch are dashed.

The important feature to grasp in figure 6 is that fixing ϵ and increasing N corresponds to moving along one of the curves: for any fixed value of ϵ, no matter how small, the heat transfer can always be made to resemble that in an isoviscous fluid by making N large enough. On the other hand, fixing N and decreasing ϵ corresponds to moving across the diagram on a horizontal line: for any fixed value of N, no matter how large, the presence of the deformation layer controls the heat losses if ϵ is small enough.

It is also worth while to note that, if $N_1 = \epsilon^{\frac{1}{2}}N$ and $D_1 = \epsilon^2 D$, then if $DN^{-3} = \infty$ (3.23) has the form $N_1 = f_1(D_1)$; experimental results for the $N(D)$ relationship can be collapsed locally onto a single curve.

Given $N(D)$ and ϵ, (3.20) determines P. Figure 7 shows the Péclet-number–drag relationship. These curves also have three legs when ϵ is small. Table 1 gives the drag law corresponding to each leg. To interpret these results, remember that the drag is determined by the pressure. In turn, the pressure is fixed by the condition that the volume flux through the upper plane $Z = 1$ must equal that through the cylindrical control surface $r = R$. Thus the layer that accepts the larger fraction of the volume flux also determines the drag. On using (2.5) it follows that the drag is controlled by the deformation layer if $\Delta \ll 1$ and by the isoviscous flow if $\Delta \gg 1$. Now consider figure 7.

Let ϵ be made small and then fixed. Consider a series of experiments; in the first, the normal force D is very large but in the subsequent experiments it is steadily reduced. Initially the normal force is so large that the deformation layer is very thin; it chokes, and all the volume flux is carried by the isoviscous fluid. Because of this, D is independent of μ_0 and t, and it depends only linearly on U. Although it is of the same order of magnitude as the drag for the corresponding isoviscous problem, it is numerically less because the presence of the deformation layer means that the normal viscous stress never contributes substantially to D. This is the Stokes limit.

When D is reduced, the knee at $D \approx \epsilon^{-\frac{4}{3}}$ is eventually reached. To the left of this knee, $\epsilon N^3 \ll 1$, and the deformation layer is broad enough to carry the entire volume flux. It determines the relationship between D and U. Equation (3.24) shows that the normal velocity at the outer edge of the thermal layer is of order unity, so that the thickness l of the deformation layer is κ/Ut. Since the pressure is of order $\mu_0 R^2 U/l^3$, the drag is of order $\mu_0 R^4 U/l^3$. That is

$$D' \approx \frac{\mu_0 R^4 t^3 U^4}{\kappa^3}.$$

This depends strongly on U and R; increasing U makes the deformation layer thinner, and increasing R means that more volume flux passes through it. This is the lubrication limit.

As D is reduced further, the flow eventually reaches the knee at $D \approx 1$. To the left of this knee forced convection is no longer important, and D depends linearly on U because $l \approx a/t$. This is the conduction limit. As in the lubrication limit, the D increases with t, because making t larger makes the deformation layer narrower.

Note that if $P \lesssim 1$, the flow is always dominated by the viscosity contrast. For the flow to show essentially isoviscous behaviour, P must be large enough to choke the deformation layer.

In §2 we noted that for a given D the sphere adopts the faster of the two strategies available to it. This is a useful result, and it is worth understanding its limitations. Table 1 shows that, at the transition from the lubrication mechanism to the Stokes mechanism, $N \approx P$. The ratio of the drag laws for the lubrication and Stokes mechanisms is of order ϵN^3, and since the transition occurs for $\epsilon N^3 \approx 1$ the sphere does indeed adopt the strategy with the smaller drag. This can also be seen from figure 7, for near the right knee the $P(D)$ curve lies above its two asymptotes. Note that the converse is true at the left knee.

This is not too surprising. Remember that if the viscosity is a given function of position, the usual derivation of the minimum-dissipation principle holds. Table 1 shows that the $N(D)$ relationship does not change at the right knee: the flow has no choice of viscosity profile there. The knee exists only because it becomes relatively easy to deform the isoviscous fluid when D is large. The dissipation principle leads us to expect that, if the choice of two strategies does not involve changing the viscosity profile, the flow will adopt the strategy with the smaller drag. This happens at the right knee. However, the left knee in the curve exists only because there is a change in the dominant mode of heat transfer; it has nothing to do with the Navier–Stokes equations. There is no reason to expect the transition to be governed by the minimum-dissipation principle, and it is not. In saying that the sphere adopts the faster of the two strategies available to it, we refer only to the transition from the lubrication to the Stokes mechanism. This result is used in §5.

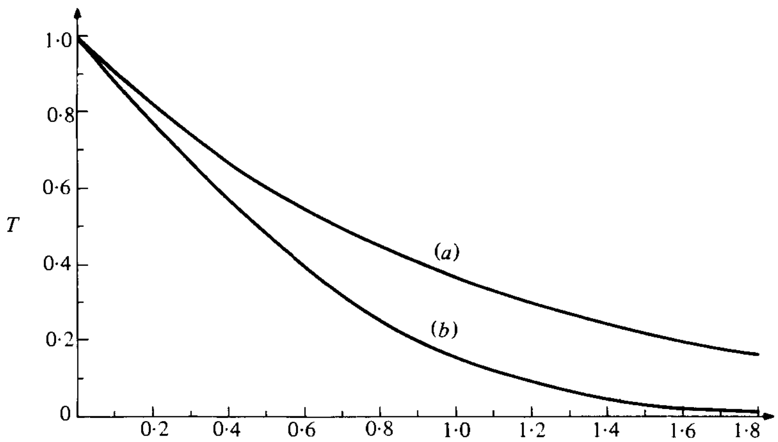

FIGURE 8. Temperature profile for the hot stagnation-point flow. The horizontal scale for the lubrication limit (*a*) is z; for the Stokes limit (*b*) it is $z(\frac{4}{3}P)^{\frac{1}{2}}$.

Figure 5 shows the tangential velocity in these two extremes. In fact the tangential velocity profile qualitatively resembles that in figure 5(*a*) whenever $\Delta = \epsilon N^3 \ll a/l$, even though the drag is qualitatively isoviscous for $\Delta \gtrsim 1$; (2.5) and (2.6) make this clear for the case $R = a$. This is surprising at first sight, but it must be so because the drag is determined by the layer that carries the most volume flux and not simply by the layer in which the velocity is greatest.

Although the curves in figure 7 qualitatively resemble those for the Nusselt number of figure 6, there is one important difference. In the graph for N, the right knee occurs for $D \approx \epsilon^{-\frac{4}{3}}$. If ϵ is very small, there is a substantial range, $\epsilon^{-\frac{4}{3}} \lesssim D \lesssim \epsilon^{-2}$, in which the drag is essentially isoviscous, but in which the heat losses are controlled by the ability of the deformation layer to bring cold fluid next to the hot plane. Of course the heat flux is much less in this transitional case that it is in the lubrication limit, because in the lubrication limit the jump in normal velocity is U, but in the transitional case it is much less than U (see (2.5)).

This effect means that the $N(P)$ curve has three knees. For this reason I have not emphasized the $N(P)$ relationship by including a graph. In making this point, I am thinking of a recent experiment by Ribe (1982), which we discuss in §3.4. The essential point is that only moderate values of t (less than 10) can be obtained experimentally. The knees in the $N(P)$ curves are then separated by only a modest range of P, and it will be much easier to pick the essential transitions out of the scatter of the experimental data if that data is presented in the form shown in figures 6 and 7; those curves have only two structures to be resolved.

Table 1 also gives the asymptotic forms for the $N(P)$ relationship. Numerical values for the $N(P)$ relationship can be obtained by using figures 6 and 7 together.

For completeness, figure 8 shows the temperature profile in the two extremes $D \ll \epsilon^{-\frac{4}{3}}$ and $D \gg \epsilon^{-2}$.

Consider two further practical points. First, in the limit $t \to \infty$ the adjustment layer is negligible. We need to know how large t should be for this to be true in practice. I estimated the value of t necessary by integrating the momentum equation (3.6) to give G_0 as a function of T. In the lubrication limit $T \sim e^{-z}$, and it is possible to estimate the contribution of each of the three layers to the integral that determines G_0. Using rather conservative bounds I found that if $t > 10$ the contribution from the deformation layer dominates that from the adjustment layer. It would be helpful to have a numerical calculation for comparison.

Secondly, it follows from (3.13) that the dimensional thickness of the deformation layer is

$$l' = \frac{1}{|\gamma T_z(0)|}. \tag{3.25}$$

Most of the applications involve replacing a viscosity profile of the form shown in figure 1 with the form $\nu = A\,e^{-\gamma T}$. The entire asymptotic analysis shows the essential viscosity variations to occur within the deformation layer, so that the viscosity profile must be fitted accurately there. This can be done by taking

$$\gamma = \frac{d}{dT_0}\ln \nu(T_0), \tag{3.26}$$

where T_0 is the temperature at the hot surface. Since the dimensional boundary-layer thickness

$$\delta' = \frac{\Delta T}{|T_z(0)|},$$

it follows that $l'/\delta' \rightarrow 0$ if $t = \gamma \Delta T \rightarrow \infty$; the essential point is that the relevant value of t is based on the local value of γ and the total temperature drop ΔT. For the asymptotic theory to be valid this value of t must be bigger than 10.

It is important to recognize that there is in general no connection between the local value of t and ν_0/ν_∞. Most of the fluids that experimentalists use to model variable-viscosity flows have the unfortunate property that ν_0/ν_∞ can be made very small by making ΔT large, while the local value of t rarely exceeds 5 or 6. This is not big enough; in the geophysical flow it is easy to have $t \geqslant 10$ even if there is no partial melting of the rock surrounding the magma. This suggests that asymptotic analysis on numerical simulation is more likely to bring out the essential features of these flows than experiments are.

So far we have concentrated on the essential case $f_0'' = 0$. The lower plane is rigid ($f_0' = 0$) in polymer moulding (Lee *et al.* 1982). In addition, any experimental modelling of the magma problem will be done with rigid spheres, although the condition $f_0'' = 0$ is most appropriate. Thus it is useful to point out the special effects due to the rigid surface, so that they do not confuse the issue.

3.4. *Behaviour when the lower plane is rigid*, $f_0' = 0$

Consider the differences between this case and the previous one. The essential point is that if ν_0/ν_∞ is made very small, and then fixed, it is possible to make a/l so large that the external isoviscous flow sees the lower plane as rigid. If a/l is fixed at this value and ν_0/ν_∞ is decreased, the first effect of the viscosity contrast will be to allow the external flow to slip past the rigid plane. This results in a Couette flow in the deformation layer, and the effect becomes important when the jump in tangential velocity across the Couette flow is comparable to U. This occurs when $(a/l)\,(\mu_0/\mu_\infty) \approx 1$. Thus if $\Delta \gg (a/l)^2$ the drag is given by the isoviscous result for a rigid surface and the Nusselt number $N \propto P^{\frac{1}{3}}$. But if $\Delta \ll (a/l)^2$ the external flow sees the lower plane as stress-free, and our earlier discussion for the stress-free case holds qualitatively. There is a very slight numerical difference because of the different boundary condition on the flow in the deformation layer.

Notice that for $t^4\epsilon^{-3} \ll D$ the drag law is simply that for the Reynolds problem (see Batchelor 1967, p. 228). For $\epsilon^{-\frac{4}{3}} \ll D \ll t^4\epsilon^{-3}$ the external flow sees the lower plane

	Drag, D'	$N(D)$	$N(P)$
$0 \leqslant D \ll 1$ (Conduction limit)	$\frac{1}{8}\pi\mu_0 \frac{R^4}{a^3} t^3 U$	1	
$1 \ll D \ll \epsilon^{-\frac{4}{3}}$ (Lubrication limit)	$\frac{1}{8}\pi\mu_0 \frac{R^4 t^3}{\kappa^3} U^4$	$(\frac{1}{2}D)^{\frac{1}{4}}$	$\dfrac{P}{1-e^{-P}}$
$\epsilon^{-\frac{4}{3}} \ll D \ll \epsilon^{-2}$	$\frac{3}{8}\pi\mu_\infty \frac{R^4}{a^3} U$		
$\epsilon^{-2} \ll D \ll t^4\epsilon^{-3}$ (Stokes limit)		$\left(\dfrac{\epsilon D}{2\pi}\right)^{\frac{1}{2}}$	$\left(\dfrac{3}{\pi}\right)^{\frac{1}{2}} P^{\frac{1}{2}}$
$t^4\epsilon^{-3} \ll D$	$\frac{3}{2}\pi\mu_\infty \frac{R^4}{a^3} U$	$\left(\dfrac{t^2 D}{2\pi}\right)^{\frac{1}{3}}$	$\left(\dfrac{12 e^t}{\pi t}\right)^{\frac{1}{3}} P^{\frac{1}{3}}$

TABLE 2. Asymptotic forms for the case $f_0' = 0$ (rigid lower plane)

as traction-free, and the drag law is the same as that given in table 1. Finally, if $1 \ll D \ll \epsilon^{-\frac{4}{3}}$ the deformation layer controls the drag; table 2 shows that if $f_0' = 0$ then D' is twice its value for a traction-free plane.

When $\epsilon^{-2} \ll D \ll t^4\epsilon^{-3}$, the $N(P)$ relationship is the same as it is for the case $f_0'' = 0$, and it is in fact the same as that for the corresponding isoviscous problem. But if $t^4\epsilon^{-3} \ll D$, table 2 shows that $N \propto P^{\frac{1}{3}}$ when the lower plane is rigid. This is qualitatively the same as the result for the corresponding isoviscous problem. But even at these large values of a/l the present result still contains the viscosity contrast. This is interesting, for it means that the fine details of our flow are always influenced by the viscosity contrast, even if the flow as a whole eventually manages to imitate the gross properties of an isoviscous flow.

Ribe (1982) has recently made an interesting and very careful experimental study of the problem. He kept the temperature of a hot metal ball constant as it fell through cold golden syrup. Ribe plots both N and P as a function of D; he uses the viscosity contrast rather than t as the parameter. The transition from $N \propto P^{\frac{1}{2}}$ to $N \propto P^{\frac{1}{3}}$ is particularly marked, but none of the other knees in the $N(P)$ curve is resolved. I think that this is so in part because t was typically less than 5 or 6, and in part because Ribe presents data for many different viscosity contrasts on the same graph. If $t = 5$, $\epsilon \approx 0{\cdot}88$, and the knees at $D \approx 1$ and $D \approx \epsilon^{-2}$ merge. If t were actually this small in the geophysical problem, the fine distinctions of the asymptotic theory would be a waste of time. In practice $t > 10$, so that $\epsilon < 0{\cdot}05$ and knees are quite distinct (see figure 6). We return to this in §5.

Table 2 shows that the external flow begins to slip freely over the lower plane when $D \approx t^4\epsilon^{-3}$, so that $N \approx e^t/t$. This value of N is large when t is. For example, if $t = 10$, $e^t/t \approx 10^3$; the boundary layer must be very thin to choke the deformation layer completely.

So far we have neglected the complications introduced by the shape of the sphere in order to bring out the main changes that occur in the flow when D is varied with $t \gg 1$. There is no simple exact solution for the sphere that is valid for all D. However, it is possible to find separate asymptotic solutions in the lubrication limit and in the Stokes limit. We give the solution in the first case.

4. The hot sphere for arbitrary P in the lubrication limit $\epsilon^{\frac{4}{3}}D \to 0$

This analysis corresponds to studying the left and middle legs in figures 6 and 7. In addition we estimate the location of the right knee in the drag law by using the fact that the sphere adopts the faster of the two mechanisms available to it.

As before we give an asymptotic solution for the profile $\nu = Ae^{-\gamma T}$ and $t \to \infty$.

Consider the steady creeping flow shown in figure 9. The tangential component of stress vanishes on the surface of the sphere. For large t the width l of the deformation layer is small compared with the thermal lengthscale δ. We shall find both the heat flux out of, and the drag on, the sphere to be determined by those parts of the flow for which $\delta \lesssim a$. Hence $l \ll a$, so that within the deformation layer we can neglect both the effects of curvature and the streamwise diffusion of momentum.

Let a circumflex denote those dimensionless quantities that are $O(1)$ within the deformation layer, and let a tilde denote those that are $O(1)$ over distances of order a.

Let

$$\mu = \mu_0 \hat{\mu}, \quad \tilde{T} = \frac{T - T_\infty}{\Delta T} = 1 - \frac{1}{t}\hat{T}, \quad p = \hat{p}\frac{\mu_0 a^2}{l_0^3}U.$$

Also let

$$u_\theta = \frac{a}{l_0}U\hat{u}_\theta, \quad u_r = U\hat{u}_r, \quad r = a\tilde{r} = a + l_0\hat{r}.$$

From now on we shall delete the circumflexes from dimensionless quantities, but will retain the tilde. On letting $l_0/a \to 0$ with P fixed, the dimensionless governing equations become

$$\frac{\partial u_r}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) = 0, \tag{4.1}$$

$$\frac{\partial p}{\partial r} = 0, \quad \frac{\partial p}{\partial \theta} = \frac{\partial}{\partial r}\left(\mu\frac{\partial u_\theta}{\partial r}\right), \qquad (4.2), (4.3)$$

$$\mu = e^T, \quad \frac{\partial^2 T}{\partial r^2} = 0; \qquad (4.4), (4.5)$$

once again the temperature gradient is constant within the deformation layer. These are to be integrated subject to the conditions that $T = 0 = u_r = \partial u_\theta/\partial r$ on $r = 0$, and that $u_\theta \to 0$ and $u_r \to -\cos\theta$ as $r \to \infty$. Note that $u_\theta \to 0$ as $r \to \infty$ because the tangential velocity is of order $(a/l)U$ within the deformation layer, whereas it is only of order U in the external flow.

From equation (4.5) it follows that

$$T = rK(\theta), \tag{4.6}$$

where the arbitrary function $K(\theta)$ is fixed by requiring that (4.6) match to the solution of the external energy equation. Thus $\mu = e^{rK(\theta)}$. Now the expressions

$$u_r = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial r} \tag{4.7}$$

satisfy the equation of continuity (4.1) exactly. Let $\zeta = rK(\theta)$. The momentum equation (4.3) then admits a similarity solution of the form

$$\psi = \tfrac{1}{2}f(\zeta)\sin^2\theta,$$

where the velocity profile

$$f' = \tfrac{1}{2}(1+\zeta)e^{-\zeta}, \quad f = 1 - (1 + \tfrac{1}{2}\zeta)e^{-\zeta}, \tag{4.8}$$

and the pressure gradient

$$\frac{\partial p}{\partial\theta} = -\tfrac{1}{4}K^3(\theta)\sin\theta. \tag{4.9}$$

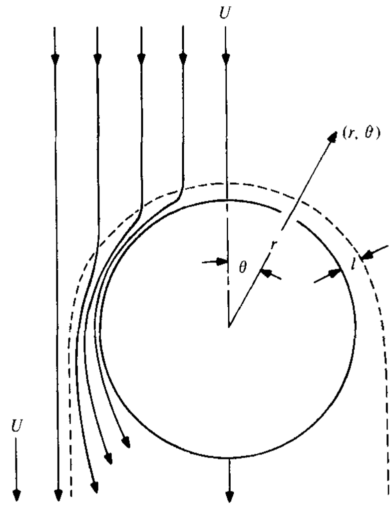

FIGURE 9. Flow past a hot sphere in the lubrication limit.

The velocity profile (4.8) is necessarily the same as that for the stagnation-point flow (see figure 5*b*). The dimensional thickness of the deformation layer is $l_0/K(\theta)$. $K(\theta)$ is determined by solving the energy equation in the uniform external flow.

We have

$$-\cos\theta\frac{\partial \tilde{T}}{\partial \tilde{r}}+\frac{\sin\theta}{\tilde{r}}\frac{\partial \tilde{T}}{\partial \theta}=P^{-1}\nabla^2\tilde{T}. \tag{4.10}$$

This is the Oseen equation. Its general solution is given by Illingworth (1963, p. 192), but we consider only the extreme cases $P\to 0$ and $P\to\infty$. In each case the solution can be obtained by approximating (4.10) directly.

If $P\to 0$, the solution is $\tilde{T}=1/\tilde{r}$. Matching this to the temperature field within the deformation layer shows that $K(\theta)=l_0 t/a$, so that the dimensional thickness of the deformation layer is a/t; it is independent of θ because the heat is transferred by conduction alone.

Given $K(\theta)$ we can find the drag due to both hemispheres. It is

$$D' \sim \tfrac{1}{3}\pi\mu_0 Uat^3, \quad P\to 0. \tag{4.11}$$

This is about five times the corresponding result (with $R=a$) in table 1.

It is interesting to compare this with the drag given by Stokes' law for flow past a traction-free sphere, namely $4\pi\mu_\infty aU$. The ratio of the two is $\frac{1}{12}t^3e^{-t}$. If $t=10$ this is $2{\cdot}0\times10^{-3}$: lubrication is cost-effective. This also gives an interpretation of the parameter ϵ.

If $P\gg 1$ but the external fluid is still rigid, the isotherm $T=T_0-\gamma^{-1}$ separates from the sphere at the equator, as shown in figure 9, and closes on itself only far behind the sphere. The sketch shows that behind the sphere the width of the deformation layer is comparable to the radius a. Hence the tangential velocity within it is comparable to U, and it follows that the contribution of the lower hemisphere to the drag is far less than that of the upper hemisphere. Similarly, the heat flux out of the sphere is also controlled by the upper hemisphere.

Let $\tilde{r}=1+r_2/P$, so that r_2 is of order unity within the thermal boundary layer. Then, in the limit $P\to\infty$,

$$-\cos\theta\frac{\partial \tilde{T}}{\partial r_2}=\frac{\partial^2 \tilde{T}}{\partial r_2^2}. \tag{4.12}$$

Hence
$$\tilde{T} = \exp\{-(\tilde{r}-1)P\cos\theta\}, \tag{4.13}$$

so that
$$K(\theta) = \frac{l_0 U t}{\kappa}\cos\theta. \tag{4.14}$$

The drag on the forward hemisphere is
$$D' \sim \frac{\pi a^4 \mu_0 U^4}{48\kappa^3} t^3, \quad P \to \infty. \tag{4.15}$$

The transition from (4.11) to (4.15) occurs for $P \approx 1$ and corresponds to the lowest knee in figure 7. We can estimate the location of the second knee in the following way. The ratio of (4.15) to the drag given by Stokes law is $\frac{1}{192}\epsilon P^3$, where $\epsilon = t^3(\nu_0/\nu_\infty)$, as in §3. Provided that $P \ll (192/\epsilon)^{\frac{1}{3}}$ the drag D is smaller than that given by Stokes law, but for $P \gg (192/\epsilon)^{\frac{1}{3}}$ the sphere can move most quickly by deforming the external fluid. If $t = 10$, $(192/\epsilon)^{\frac{1}{3}} = 35$. Thus there is a substantial range of P, $1 < P < 35$, in which the drag is given by (4.15). In practice, the critical Péclet number will be slightly less than this, for even in the Stokes limit the drag on the sphere will be a little less than that for the corresponding isoviscous flow.

According to (4.15) the drag is only one-third of the estimate given by the stagnation-point flow (with $R = a$). This must be the case because the width of the layer increases with θ.

The average Nusselt number based on the forward hemisphere is
$$N \sim \tfrac{1}{2}P, \quad P \to \infty. \tag{4.16}$$

The thermal-boundary-layer thickness
$$\frac{\delta}{a} = P^{-1}\frac{1}{\cos\theta},$$

and becomes infinite as $\theta \to \frac{1}{2}\pi$. The singularity occurs because the radial velocity changes sign at the equator, so that the thermal boundary layer is maintained there by the tangential advection of heat. Balancing tangential advection against radial diffusion shows that $\delta/a \approx P^{-\frac{1}{2}}$, $|\frac{1}{2}\pi - \theta| \leqslant P^{-\frac{1}{2}}$. Behind the sphere this result also breaks down and δ becomes comparable to a (see figure 9). Thus the correction to (4.16) due to the equator and rear hemisphere is only of order unity.

The isotherms are closed by diffusion far behind the sphere. Notice how different this is from the case in which the thermal layer is controlled by the isoviscous flow: in that case $\delta/a \ll 1$ everywhere except at the trailing stagnation point.

5. Application to the geophysical problem

5.1. *Terminal speed*

Suppose now that the sphere is less dense than the surrounding fluid by a fixed amount $\Delta\rho$. The terminal speed U follows by setting the dimensional drag D' equal to the total buoyancy $\frac{4}{3}\pi a^3 g\Delta\rho$ of the sphere. Let $g' = g(\Delta\rho/\rho_\infty)$. Then the results in §4 give
$$\left.\begin{aligned} U &\sim 4\frac{g' a^2}{\nu_0}t^{-3} && (P \to 0) \\ U &\sim 2\sqrt{2}\left(\frac{g'\kappa^3}{a\nu_0}\right)^{\frac{1}{4}} t^{-\frac{3}{4}} && (P \to \infty) \end{aligned}\right\} \Delta \to 0. \tag{5.1}$$

For other values of Δ the terminal speed can be estimated from figure 7 by setting $D' = \frac{4}{3}\pi a^3 g \Delta\rho$; that is

$$D = \frac{64}{3}\frac{g' a^3}{\nu_0 \kappa} t^{-3}. \tag{5.2}$$

There are two interesting aspects to the second form of (5.1). First, the terminal speed depends only weakly on the poorly known quantities a and ν_0. Secondly, U decreases as the radius a increases: notice that this is not true in the conduction limit. To see why this occurs, note that for the sphere to advance through a distance a it must push a volume πa^3 of fluid through the deformation layer. In the conduction limit, the width of this layer is proportional to a, but in the limit $P \to \infty$ it depends only weakly on a:

$$l \sim \frac{1}{2\sqrt{2}}\left(\frac{a\nu_0\kappa}{g'}\right)^{\frac{1}{4}} t^{-\frac{1}{4}}. \tag{5.3}$$

This effect means that when $P \gg 1$ the drag increases more rapidly with a than the total buoyancy does (see (4.15)), and the terminal speed therefore decreases as a increases.

The geophysical values of a, ν_0 and t are uncertain, but at least at high Péclet numbers the results for U and l do not depend strongly on them. Marsh (1978) uses $a = 1$ km, based partly on the size of plutons exposed at the Earth's surface and partly on estimates of the total volume of magma that makes up an island-arc volcano. Next, the viscosity of the country rock near its solidus is poorly known, but is unlikely to be less than 10^{16} cm^2 s^{-1}. Lastly, one estimate for t can be got by using the results of Ashby & Verrall (1978) for olivine; this estimate is a low one because Ashby & Verall's work is for solid-state creep and therefore does not include the possibility of partial melting. This gives $t \approx 10$ (see §1). Other estimates for t follow from the measurements by Sakuma (1953) on the viscosity of lava in its melting range. I estimated that immediately above the solidus $\gamma \approx 2{\cdot}5 \times 10^{-2}\,\mathrm{K}^{-1}$ and that in the middle of the melting range $\gamma \approx 4 \times 10^{-1}\,\mathrm{K}^{-1}$. If $\Delta T = 500$ K this gives $t \approx 12$ and $t \approx 200$ respectively. The larger value is not likely to be relevant.

Thus we can get a rough idea of the speed of ascent by supposing that $t = 20$, $g' = 300$ cm/s^2, $\kappa = 10^{-2}$ cm^2/s, $a = 10^5$ cm and $\nu_0 = 10^{16}$ cm^2/s. Then $U \approx 3 \times 10^{-7}$ cm/s, $\delta \approx 400$ m, $l \approx 20$ m and $\Delta \approx 0{\cdot}1$ if $\nu_\infty = 10^{22}$ cm^2/s. This estimate for U is quite sturdy, as we shall now see.

In practice, the magma probably begins its ascent at the same temperature as the surrounding fluid. The driving stress is of order $ag\Delta\rho$, and is large enough for the surrounding rock to deform as a power-law fluid rather than as a Newtonian one. As the sphere rises through the geothermal boundary layer, its surroundings become colder and stiffer. Thus, if the sphere ascends at constant temperature T_0, it eventually reaches a level at which the effective viscosity contrast is large enough for the lubrication mechanism to give a faster terminal speed than the Stokes mechanism (Morris 1980). The analysis in §3 shows that the sphere adopts the faster of the two strategies. This should also be true for a power-law fluid. Therefore we can estimate the depth of transition by first calculating a terminal velocity for a sphere that rises through a power-law fluid without heating it, and then comparing that velocity with the terminal velocity of a sphere rising by the softening mechanism. Table 3 shows the bounds on the terminal velocity for the isothermal problem; I calculated them using the upper and lower bounds on the drag given by Wasserman & Slattery (1964) and assuming that $a = 5$ km with a chemically imposed density difference $\Delta\rho = 0{\cdot}6$ g cm^{-3}.

Table 4 shows the terminal velocity for a sphere ascending by the softening

Depth (km)	p_∞ (kbar)	T_∞ (K)	U_{max} (cm s^{-1})	U_{min} (cm s^{-1})
20	5	500	10^{-39}	10^{-39}
50	16	1300	2×10^{-7}	1×10^{-8}
100	32	1600	5×10^{-4}	2×10^{-5}

TABLE 3. Stokes limit in a power-law fluid

Depth (km)	$3t$	P	U (cm s^{-1})	l (m)
20	23·9	16·8	3×10^{-7}	10
50	10·4	20·6	4×10^{-7}	20
100	5·4	31·5	6×10^{-7}	30

TABLE 4. The lubrication limit in a power-law fluid; $T_0 = 1900$ K

mechanism. The appendix gives the drag law in this case. The drag law is valid when $3t \gg 1$. As in table 3, $a = 5$ km and $\Delta\rho = 0{\cdot}6$ g cm^{-3}.

The value of T_0 used in table 4 is high. This compensates for the fact that the constitutive relationship does not include the effects of partial melting. If $T_0 = 1400$ K and there is no partial melting, then $U = 2\times10^{-8}$ cm s^{-1} at 20 km.

Comparing the tables makes it clear that, if $a = 5$ km, the ascent begins by the first, essentially isothermal, mechanism. The transition to the lubrication limit occurs at about 50 km, and the terminal speed in the lubrication limit is (once again) about 10^{-7} cm s^{-1}. This makes the response time for the island arc about 10^6 years.

The details in the appendix show that the terminal speeds in table 4 depend only weakly on a, whereas those in table 3 depend on the fourth power of a. This means that for smaller spheres the transition occurs much deeper in the earth.

5.2. *Energy constraints*

If the softening mechanism applies, the response time for the island arc meets the geological constraint; to that extent Marsh's suggestion works. Now we must consider the price of success. The essential point is that for the sphere to advance through one of its own radii by the softening mechanism, it must soften a volume πa^3 of country rock. Since the specific heat of the magma is comparable to that of its surroundings, an isolated sphere (with no internal heat source) would be in rough thermal equilibrium with its surroundings as it rose, and would solidify. We can quantify this effect. In this section t is time. Initially suppose that T_∞ is constant.

Let the subscript p refer to the hot parcel, and let the subscript a refer to the ambient fluid. Then the total heat flux out of the sphere is $(\rho C_p)_a \kappa \Delta T 4\pi a N$, where N is the average Nusselt number and C_p is the specific heat at constant pressure. An energy balance applied to the sphere then gives

$$\frac{dT}{dt}+\frac{1}{\tau}(T-T_\infty) = 0, \tag{5.5}$$

where the cooling time τ is given by

$$\frac{U\tau}{a} = \frac{(\rho C_p)_p}{3(\rho C_p)_a}\frac{P}{N}. \tag{5.6}$$

Note that a/U is the time taken by the sphere to travel through one of its own radii. The quantity P/N can be estimated from the results for the squeezing flow (table 1); it increases steadily as the driving force D (and hence P) increases. Intuition tells us that if $P \to 0$ the body loses heat quickly by conduction but travels only slowly. Indeed, (5.6) says that $U\tau/a \to 0$ in the conduction limit. Next, in the lubrication limit $P/N \to 1$, and the sphere cools substantially by the time it has travelled through one of its own radii. Lastly, if $P \to \infty$ conductive losses ought to be very small and the body should ascend nearly adiabatically. Note that in this limit the viscosity variations affect neither the drag nor the heat flux. Table 1 shows that in the Stokes limit $P/N \approx P^{\frac{1}{2}}$. Since $P \gg 1$ whenever the Stokes limit applies, the sphere travels through many of its own radii before cooling to the temperature T_∞ of its surroundings. We should expect this. When the lubrication mechanism applies, the body moves by softening the column ahead of it; but, when the Stokes mechanism applies, the body shoulders the fluid aside and does not heat it as effectively.

It is essential to recognize that whilst the lubrication mechanism results in fairly large Péclet numbers (table 4), it does so only by thermally mixing the body with its surroundings. Sometimes it is suggested that large viscosity variations allow hot material to ascend nearly adiabatically, but this idea is simply not borne out by the calculations.

To emphasize this point, consider a hot sphere rising steadily through the geothermal boundary layer. It sees T_∞ varying in time. Suppose for example that the geotherm is exponential:

$$T_\infty(z) - T_\mathrm{m} = (T_\mathrm{s} - T_\mathrm{m})\, e^{-z/L},$$

where T_m is the temperature at the heart of the mantle, T_s is the temperature at the Earth's surface ($z = 0$), and L is the thickness of the geothermal boundary layer. Then the solution of (5.5) is

$$T(t) - T_\infty(t) = \left\{T(0) - T_\infty(0) - \left(1 + \frac{L}{U\tau}\right)^{-1} (T_\mathrm{m} - T_\mathrm{s})\right\} e^{-t/\tau} + \left(1 + \frac{L}{U\tau}\right)^{-1} (T_\mathrm{m} - T_\infty(t)). \tag{5.7}$$

Let $(\rho C_p)_\mathrm{p}/(\rho C_p)_\mathrm{a} = 1$, let $T_\mathrm{m} - T_\mathrm{s} = 1400$ K, and suppose that the lubrication mechanism begins to work at $t = 0$ when $T - T_\infty = 500$ K. In the lubrication limit (4.16) gives $N = \frac{1}{2}P$, so that $U\tau/a = \frac{2}{3}$ and $L/U\tau = 3L/2a$. Let $L = 30$ km. It follows that when $t = 2a/U$ (so that the body has travelled through one of its own diameters) the first term in (5.7) is about 18 K if $a = 5$ km, and 25 K if $a = 1$ km. The second term is biggest at the Earth's surface, where $T_\mathrm{m} - T_\infty(t) = 1400$ K in this example; the second term is 140 K if $a = 5$ km, and about 30 K if $a = 1$ km. It follows that, if the isolated sphere is small enough to see T_∞ varying only over a time much longer than τ, then the rising sphere is in thermal equilibrium with its surroundings.

This is a nuisance for an individual parcel, as it quickly solidifies. Yet for the diapir as a whole, it is a very effective method of softening and removing the highly viscous country rock. A second parcel, following before the thermal anomaly due to the first has diffused, rises through low-viscosity fluid until it reaches the cold leading edge of the diapir. There it begins to penetrate cold fresh rock by the softening mechanism. It soon dumps all its heat into the leading edge of the diapir, and equilibrates with the cold rock ahead of it. The process repeats itself. It provides an effective mechanism for getting heat from the source region into the pathway, where it is needed. Given a rapid supply of spheres, the leading edge of the diapir propagates through the lithosphere at a speed of order 10^{-7} cm s^{-1}. In contrast, an individual sphere following the Stokes mechanism could travel much farther before solidifying, but an enormous stress would be necessary to drive it at this speed. Marsh (1982) describes some additional features of the process.

To open a passage to the Earth's surface by the softening mechanism a column of cross-sectional area πa^2 must be softened. If the mechanism is to apply over a depth D, the smallest amount of energy necessary is

$$\pi a^2(\rho C_p)_{\mathrm{a}} \int_0^D dz\,(T_{\mathrm{p}}-T_\infty),$$

where T_{p} is the temperature of the parcel at depth z. If the geotherm is exponential with a boundary-layer thickness of 30 km, and T_{p} is 1500 K, than the amount of energy needed is about 10^{27} erg if a is 1 km. This is not a trivial amount of energy. For example a sphere of rock with radius 1 km releases about 10^{26} erg if it cools through 1000 K. About ten of these spheres would be necessary. Even if we allow for the release of gravitational potential energy this qualitative picture does not change. Thus, many spheres must follow the same path before one can reach the surface.

I think that further numerical work for a single sphere will not change the picture drawn in §3. It would be helpful to have a more accurate study of the flow in a power-law fluid, but above all a numerical study of the interaction of successive parcels would be useful. It would be enough to make a model with two finite viscosities; one to represent the magma and the other to represent the deformation layer.

There is a final point. The processes we have discussed are incompressible; the magma advances by softening the material in front of it, and then exchanging positions with it. Under some conditions, magma can be transported by fracturing the rock through which it passes. If the stresses associated with plate tectonics are such as to open cracks in the country rock, then this is a plausible mechanism. But an island arc must be under compression because the oceanic plate is being pushed against it. This suggests that a fracture can only exist if the magma has enough potential energy either to lift aside the material that is in its way, or else to push it aside by compressing it. The second mechanism is implausible because the bulk modulus of rock is very large, and the first mechanism can apply only near the Earth's surface. It would be interesting to study the energetics of these two processes, taking into account the compressibility of the magma.

6. Summary

We considered the problem of finding the response time for a island arc; that is the time needed to open a fresh pathway from the magma source to the volcano. We studied a softening model due to Marsh (1978) in detail. The simple model of squeezing flow between parallel planes (§3) captures the essential fluid dynamics, and we gave a detailed discussion of it for a Newtonian fluid of strongly temperature-dependent viscosity. We stressed two facts. First, when the Péclet number is large, the problem has two large parameters, namely P and $\gamma \Delta T$. Depending on the relationship between them, the flow can either be qualitatively isoviscous, or dominated by the viscosity contrast. Secondly, when modelling the geophysical flow experimentally, it is not enough to make the viscosity ratio μ_∞/μ_0 large. Rather, the value of t at the hot surface must be large. In §5 we extended the discussion to the more realistic case of a power-law fluid, and applied it to the geophysical problem. We found the softening mechanism to give a response time of less than a million years, at a cost of somewhat greater than 10^{27} erg. During the first 50 km the magma follows a qualitatively isoviscous mechanism; but in the remaining 50 km it follows Marsh's advice and ascends by lowering the viscosity of the country rock.

This work contains the essential results in chapter 2 of Morris (1980). I would like to thank B. D. Marsh and O. M. Phillips for getting me started; G. M. Corcos, P. J. Pagni and two referees for stopping me; and N. Ribe for a preprint of his experimental study.

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Appendix. Solution for a power-law fluid

In a power-law fluid, the shear stress σ and strain rate $\dot{e}$ are related by $\sigma = m\dot{e}^n$. This applies to rock at moderately high temperatures and deviatoric stresses (see Ashby & Verrall 1978). For olivine, Ashby & Verrall give $n \approx \frac{1}{3}$ and

$$[m(T)]^3 = \frac{kTG^2}{bA_1} \exp \frac{Q+pV}{RT}; \tag{A 1}$$

here Boltzmann's constant $k = 1{\cdot}38 \times 10^{-23}$ J K^{-1}, the rigidity $G = 8{\cdot}13 \times 10^{10}$ N m^{-2}, the Burger's vector $b = 6{\cdot}0 \times 10^{-10}$ m, $A_1 = 0{\cdot}45$, $R = 8{\cdot}314$ J K^{-1} mol^{-1}, the activation energy $Q = 522$ kJ mol^{-1} and the activation volume $V = 1{\cdot}1 \times 10^{-5}$ m^3 mol^{-1} (Stocker & Ashby 1973).

Repeating the analysis of §4 for this case gives the drag law

$$D' \sim K_n \pi a^3 G_0 m(T_0)\, t^{2n+1} \left(\frac{U}{\kappa}\right)^{2n+1} (\tfrac{1}{2}Ua)^n \quad (P, 3t \to \infty). \tag{A 2}$$

Here

$$G_0 = \left[\left(1+\frac{1}{n}\right)!\right]^{-n} n^{-2n-1}, \quad K_n = \frac{(\frac{1}{2}n+\frac{1}{2})!\, n!}{2(\frac{3}{2}(n+1))!}, \quad t = \Delta T\left|\frac{d}{dT_0} \ln m(T_0)\right|. \tag{A 3}$$

For $n = \frac{1}{3}$ the terminal speed is

$$U = 4{\cdot}9\left[\frac{g\Delta\rho\kappa^{\frac{5}{3}}}{a^{\frac{1}{3}}m(T_0)}\right]^{\frac{1}{2}} (3t)^{-\frac{5}{6}} \quad (3t \gg 1), \tag{A 4}$$

and $a/l = 3Pt$.

For the isothermal case

$$U = \frac{2}{27X^3} \frac{a^4(g\Delta\rho)^3}{[m(T_\infty)]^3}. \tag{A 5}$$

X is a numerical coefficient of order unity; for a solid sphere Wasserman & Slattery (1964) give $0{\cdot}53 \leqslant X \leqslant 1{\cdot}62$. No bounds have been published for a traction-free sphere, but Nakano & Tien (1968) have calculated an approximate value of X that lies between the bounds for a solid sphere given by Wasserman & Slattery.

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